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COMMENT

Large N expansion for Hulthèn potential

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Abstract. A solution of the Schrödinger equation with a Hulthèn potential is presented using the $1/N$ expansion technique.

The large N expansion approximation has recently received much attention in the context of non-relativistic quantum mechanics. In particular, Mlodinow and Shatz (1982, hereafter referred to as MS) have developed a systematic procedure for solving the Schrödinger equation to successive orders in the parameter $1/N$, where N is the spatial dimension and is set equal to the physical value 3 at the end of the calculations. Their methodology has been applied to the case of the Yukawa potential by Moreno and Zepeda (1984) and Chatterjee (1985). In this comment, we study the Hulthèn potential using the MS procedure. This potential, apart from its initial interest as a possible form of nuclear interaction, has recently been shown to be a judicious choice of starting point for the perturbation theoretic treatment of screened Coulomb potentials (Dutt *et al* 1985). Furthermore, the Hulthèn potential is exactly solvable for the $l=0$ states, thereby providing a consistency check for the $1/N$ expansion technique; such checks are desirable because the mathematical foundations of the scheme remain under investigation (Mlodinow and Papanicolaou 1980, Papanicolaou 1981).

Following the MS notation closely, we write the N-dimensional radial Schrödinger equation in units $\hbar=m=1$ as

$$\left[-\frac{1}{2}\left(\frac{d^2}{dr^2}+\frac{N-1}{r}\frac{d}{dr}\right)+\frac{l(l+N-2)}{2r^2}+\tilde{V}(r)\right]\phi(r)=E\phi(r) \tag{1}$$

where $\tilde{V}(r)$ is the N-dimensional generalisation of the Hulthèn potential

$$\tilde{V}(r)=-g^2\lambda\exp[-(9/N^2)\lambda r]\{1-\exp[-(9/N^2)\lambda r]\}^{-1}. \tag{2}$$

Substituting $\psi(r)=r^{(N-1)/2}\phi(r)$, (1) becomes

$$-\frac{1}{2}\frac{d^2\psi}{dr^2}+k^2\left(\frac{(1-1/k)(1-3/k)}{8r^2}+V(r)\right)\psi=E\psi \tag{3}$$

with the definitions

$$k\equiv N+2l \tag{4}$$

and

$$V(r)\equiv(1/k^2)\tilde{V}(r). \tag{5}$$

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An expansion in inverse powers of k or N is now developed. The series for $V(r)$ is easily found to be

$$k^2V(r)=-g^2\lambda[(N^2/9\lambda r)-\tfrac{1}{2}+\tfrac{1}{12}(9/N^2)\lambda r+\tfrac{11}{80}(9/N^2)^3\lambda^3r^3-\cdots]. \qquad (6)$$

To zeroth order, therefore, $V(r)=-g^2N^2/9k^2r$, and the leading term of the eigenvalues is given by $k^2E^{(-2)}$ with

$$E^{(-2)}\equiv\frac{1}{8r_0^2}-\frac{g^2N^2}{9k^2r_0} \qquad (7)$$

where r_0 is the value for which the expression within large round brackets in (3), in the limit of large k, attains a minimum. A simple calculation yields

$$r_0=\frac{9}{4g^2}\left(\frac{k^2}{N^2}\right). \qquad (8)$$

The leading term $E^{(-2)}$ can then be rewritten as

$$E^{(-2)}=-1/8r_0^2. \qquad (7')$$

It is now convenient to subtract $k^2E^{(-2)}$ from the total energy E and define

$$\mathscr{E}\equiv E-k^2E^{(-2)} \qquad (9)$$

and

$$V_{\text{eff}}(r)\equiv\frac{1}{8r^2}-\frac{g^2N^2}{9k^2r}-E^{(-2)} \qquad (10a)$$

$$=\frac{1}{8r^2}-\frac{1}{4r_0r}+\frac{1}{8r_0^2}. \qquad (10b)$$

The reduced radial equation (3) for the Hulthèn potential is now

$$-\tfrac{1}{2}(\mathrm{d}^2\psi/\mathrm{d}r^2)+\{k^2V_{\text{eff}}(r)+(-\tfrac{1}{2}k+\tfrac{3}{8})r^{-2}-g^2\lambda[-\tfrac{1}{2}+\tfrac{1}{12}(9/N^2)\lambda r+\ldots]\}\psi=\mathscr{E}\psi. \qquad (11)$$

The treatment so far holds true for any value of angular momentum l. In order to proceed further, we have to deal separately with the ground state and excited states.

For the ground state, we take $l=0$ so that $k=N$. We also introduce a change in variable

$$x\equiv r-r_0 \qquad (12)$$

so that V_{eff} in terms of the new variable vanishes for $x=0$.

Since the ground state wavefunction is nodeless, we may write

$$\psi=\exp[U(x)]. \qquad (13)$$

Equation (11) then becomes

$$-\tfrac{1}{2}[U''(x)+(U'(x))^2]+k^2V_{\text{eff}}(x)+(-\tfrac{1}{2}k+\tfrac{3}{8})r^{-2}$$
$$-g^2\lambda[-\tfrac{1}{2}+\tfrac{1}{12}(9/k^2)\lambda r+\tfrac{11}{80}(9/k^2)^3\lambda^3r^3+\ldots]=\mathscr{E}. \qquad (14)$$

Note that, even after changing the variable from r to x, we continue to use r for simplicity, it being understood that r is replaced everywhere by $(x+r_0)$ in accordance with (12).

The wavefunction and energy are now expanded in inverse powers of k

$$U'(x) = \sum_{n=-1}^{\infty} u^{(n)}(x) k^{-n} \tag{15}$$

$$\mathscr{E} = \sum_{n=-1}^{\infty} E^{(n)} k^{-n}. \tag{16}$$

These expressions are substituted in (14) and coefficients of like powers of k are equated to obtain the recursion relations for $u^{(n)}$ and $E^{(n)}$ as follows:

$$\begin{aligned}
u^{(-1)}(x) &= -(2V_{\text{eff}}(x))^{1/2} \\
-u^{(-1)}(x)u^{(0)}(x) &= E^{(-1)} + (1/2r^2) + \tfrac{1}{2}u^{(-1)\prime}(x) \\
-u^{(-1)}(x)u^{(1)}(x) &= E^{(0)} - (3/8r^2) + \tfrac{1}{2}u^{(0)\prime}(x) + \tfrac{1}{2}(u^{(0)}(x))^2 - \tfrac{1}{2}g^2\lambda \\
&\vdots \\
-u^{(-1)}(x)u^{(n+1)}(x) & \\
&= E^{(n)} + \tfrac{1}{2}u^{(n)\prime}(x) + \tfrac{1}{2}\sum_{m=0}^{n} u^{(m)}(x)u^{(n-m)}(x) - H^{(n)}(x) \qquad (n \geqslant 1)
\end{aligned} \tag{17}$$

where $H^{(n)}(x)$ is the coefficient of k^{-n} in the series expansion for the Hulthèn potential.

The recursion relations (17) can be solved to yield $E^{(-1)}$, $E^{(0)}$, $E^{(1)}, \ldots$, etc, to any desired order. The ground state energy E can then be computed using (7′), (8), (9) and (16). We find, for instance, to order k^{-2} and k^{-3}, respectively

$$E \text{ (to order } k^{-2}) = -\tfrac{1}{2}g^4[\tfrac{716}{729} - (\lambda/g^2) + \tfrac{3}{8}(\lambda^2/g^4)] \tag{18a}$$

and

$$E \text{ (to order } k^{-3}) = -\tfrac{1}{2}g^4[\tfrac{720}{729} - (\lambda/g^2) + \tfrac{5}{16}(\lambda^2/g^4)]. \tag{18b}$$

The value of E is thus seen to rapidly approach the exact value (see e.g. Lam and Varshni 1971)

$$E_{\text{exact}} = -\tfrac{1}{2}g^4[1 - (\lambda/g^2) + \tfrac{1}{4}(\lambda^2/g^4)]. \tag{19}$$

We now turn briefly to consider the excited states. The first excited state is the 2s state for which again $l = 0$ and $k = N$. The wavefunction will have one node so that it may be written

$$\psi_{(2s)} = (x - A)\exp(U_1(x)). \tag{20}$$

This is introduced as before in (11) to obtain a differential equation in $U_1(x)$. Next, the function $U_1(x)$, the energy $\mathscr{E}_1$ and the constant A are expanded in inverse powers of k:

$$\begin{aligned}
U_1'(x) &= \sum_{n=-1}^{\infty} u_1^{(n)}(x) k^{-n} \\
\mathscr{E}_1 &= \sum_{n=-1}^{\infty} E_1^{(n)} k^{-n} \\
A &= \sum_{n=1}^{\infty} A^{(n)} k^{-n}.
\end{aligned} \tag{21}$$

Making these substitutions and equating like powers of k on both sides, we obtain the recurrence relations

$$u_1^{(-1)}(x) = -(2V_{\text{eff}}(x))^{1/2}$$

$$-xu_1^{(-1)}(x)u_1^{(0)\prime}(x) = x[E_1^{(-1)} + \tfrac{1}{2}u_1^{(-1)\prime\prime}(x) + (1/2r^2)] + u_1^{(-1)}(x)$$

$$-xu_1^{(-1)}(x)u_1^{(1)}(x) = x[E_1^{(0)} + \tfrac{1}{2}u_1^{(0)\prime}(x) + \tfrac{1}{2}(u_1^{(0)}(x))^2 - (3/8r^2) - \tfrac{1}{2}g^2\lambda]$$
$$-A^{(1)}[E_1^{(-1)} + \tfrac{1}{2}u_1^{(-1)\prime}(x) + u_1^{(-1)}(x)u_1^{(0)}(x) + (1/2r^2)] + u_1^{(0)}(x)$$

$$-xu_1^{(-1)}(x)u_1^{(2)}(x)$$
$$= x(E_1^{(1)} + \tfrac{1}{2}u_1^{(1)\prime}(x) + u_1^{(0)}(x)u_1^{(1)}(x))$$
$$-A^{(1)}[E_1^{(0)} + \tfrac{1}{2}u_1^{(0)\prime}(x) + u_1^{(-1)}(x)u_1^{(1)}(x) + \tfrac{1}{2}(u_1^{(0)}(x))^2 - (3/8r^2) - \tfrac{1}{2}g^2\lambda]$$
$$-A^{(2)}[E_1^{(-1)} + \tfrac{1}{2}u_1^{(-1)\prime}(x) + u_1^{(-1)}(x)u_1^{(0)}(x) + (1/2r^2)] + u_1^{(1)}(x)$$

$$\vdots$$

$$-xu_1^{(-1)}(x)u_1^{(n+1)}(x)$$
$$= x(E_1^{(n)} + \tfrac{1}{2}u_1^{(n)\prime}(x) + \tfrac{1}{2}\sum_{m=0}^{n} u_1^{(m)}(x)u_1^{(n-m)}(x) - H^{(n)}(x))$$
$$-A^{(1)}(E_1^{(n-1)} + \tfrac{1}{2}u_1^{(n-1)\prime}(x) + \tfrac{1}{2}\sum_{m=-1}^{n} u_1^{(m)}(x)u_1^{(n-m-1)}(x) - H^{(n-1)}(x))$$
$$\ldots - A^{(n+1)}[E_1^{(-1)} + \tfrac{1}{2}u_1^{(-1)\prime}(x) + u_1^{(-1)}(x)u_1^{(0)}(x) + (1/2r^2)]$$
$$+ u_1^{(n)}(x) \qquad (n > 1). \tag{22}$$

The solution of these recursion relations is straightforward though rather tedious. To order k^{-2}, we obtain the energy of the 2s state as

$$E_1 \text{ (to order } k^{-2}) = -\tfrac{1}{2}g^4[\tfrac{188}{729} - (\lambda/g^2) + \tfrac{3}{4}(\lambda^2/g^4)] \tag{23}$$

which compares well with the exact value (Lam and Varshni 1971)

$$E_1 \text{ (exact)} = -\tfrac{1}{2}g^4[\tfrac{1}{4} - (\lambda/g^2) + (\lambda^2/g^4)]. \tag{24}$$

Higher excited states may be treated in a similar manner though the algebra would become more involved.

To conclude, we have investigated the $1/N$ expansion approximation for a particle bound in a Hulthèn potential. The technique has been shown to work well for the ground state and the first excited state and may be extended, in a straightforward fashion, to the study of higher excited states.

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